Exercise 2

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = 1 - \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Substitute this into the integral equation.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots &= 1 - \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots)\,dt \\ &= 1 - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 - \frac{a_2}{12}x^4 - \frac{a_3}{20}x^5 - \frac{a_4}{30}x^6 - \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} a_0 &= 1 & & \\ a_1 &= 0 & & \\ a_2 &= -\frac{a_0}{2} & \rightarrow \quad & a_2 = -\frac{1}{2} \\ a_3 &= -\frac{a_1}{6} & \rightarrow \quad & a_3 = 0 \\ a_4 &= -\frac{a_2}{12} & \rightarrow \quad & a_4 = \frac{1}{24} \\ a_5 &= -\frac{a_3}{20} & \rightarrow \quad & a_5 = 0 \\ a_6 &= -\frac{a_4}{30} & \rightarrow \quad & a_6 = -\frac{1}{720} \\ &\vdots & & \vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots \\ &= \cos x. \end{aligned}$$